

Dirac Cat States in Relativistic Landau Levels

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We show that a relativistic version of Schrödinger cat states, here called *Dirac cat states*, can be built in relativistic Landau levels when an external magnetic field couples to a relativistic spin 1/2 charged particle. Under suitable initial conditions, the associated Dirac equation produces unitarily Dirac cat states involving the orbital quanta of the particle in a well defined mesoscopic regime. We demonstrate that the proposed Dirac cat states have a purely relativistic origin and cease to exist in the non-relativistic limit. In this manner, we expect to open relativistic quantum mechanics to the rich structures of quantum optics and quantum information.

Schrödinger cat states were introduced to highlight the distinctive and fundamental properties of quantum mechanics as opposed to classical theories [1]. However, their reach has gone far beyond and, presently, different subfields of quantum information, like quantum communication, fault-tolerant quantum computation, secret sharing, among others, use them as a fundamental resource [2]. For a cat state we shall understand a coherent superposition of two maximally different quantum states with the additional property of being mesoscopic. The construction of cat states based on relativistic quantum effects has not been addressed so far and it is one of the main purposes of our work. To achieve this goal, we study the Dirac Hamiltonian under specific conditions that we detail below. We want to stress that the multidisciplinary description of novel quantum relativistic effects is not only an important advance in physics fundamentals, it will also influence other physical systems that simulate their dynamics.

A relativistic electron of mass m , charge $-e$, subjected to a constant homogeneous magnetic field along the z -axis, is described by means of the Dirac equation

$$i\hbar \frac{\partial |\Psi\rangle}{\partial t} = (c\boldsymbol{\alpha}(\mathbf{p} + e\mathbf{A}) + mc^2\beta) |\Psi\rangle, \quad (1)$$

where $|\Psi\rangle$ stands for the Dirac 4-component spinor, \mathbf{p} represents the momentum operator, and c the speed of light. Here, \mathbf{A} is the vector potential related to the magnetic field through $\mathbf{B} = \nabla \wedge \mathbf{A}$, and $\beta = \text{diag}(\mathbb{I}, -\mathbb{I})$, $\alpha_j = \text{off-diag}(\sigma_j, \sigma_j)$ are the Dirac matrices in the standard representation with σ_j as the usual Pauli matrices [3]. The energy spectrum of this system is described by the relativistic Landau levels, first derived by Rabi [4]

$$E = \pm \sqrt{m^2c^4 + p_z^2c^2 + 2mc^2\hbar\omega_c(n+1)}, \quad (2)$$

where $n = 0, 1, \dots$ and $\omega_c = eB/m$ is the cyclotron frequency which describes the electron helicoidal trajectory.

In this paper, we derive an exact mapping between this relativistic model and a combination of Jaynes-Cummings (JC) and Anti-Jaynes-Cummings (AJC) interactions [5], so widely used by the Quantum Optics community. This original perspective allows a deeper understanding of relativistic effects [6], as well as the prediction of novel effects such as the existence of Dirac cat states. These paradigmatic states constitute the relativistic extension of the usual Schrödinger cat states [1]. In the same spirit as the latter, the Dirac cats involve a coherent superposition of mesoscopically distinct states, but have a purely relativistic nature.

Working in the axial gauge, where $\mathbf{A} := \frac{B}{2}[-y, x, 0]$, the relativistic Hamiltonian can be expressed as follows

$$H_D = mc^2\beta + \alpha_z p_z + c\alpha_x(p_x - m\omega y) + c\alpha_y(p_y + m\omega x), \quad (3)$$

where we have introduced $\omega := \omega_c/2$. It is convenient to introduce the chiral creation-annihilation operators

$$\begin{aligned} a_r &:= \frac{1}{\sqrt{2}}(a_x - ia_y), & a_r^\dagger &:= \frac{1}{\sqrt{2}}(a_x^\dagger + ia_y^\dagger), \\ a_l &:= \frac{1}{\sqrt{2}}(a_x + ia_y), & a_l^\dagger &:= \frac{1}{\sqrt{2}}(a_x^\dagger - ia_y^\dagger), \end{aligned} \quad (4)$$

where $a_x^\dagger, a_x, a_y^\dagger, a_y$, are the creation-annihilation operators of the harmonic oscillator $a_i^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\tilde{\Delta}}r_i - i\frac{\tilde{\Delta}}{\hbar}p_i\right)$, $i = x, y$ and $\tilde{\Delta} = \sqrt{\hbar/m\omega}$ represents the oscillator's ground state width. Let us first consider an inertial frame \mathcal{S}' which moves along the axis OZ at constant $v_z = p_z/m$ with respect to a rest frame \mathcal{S} . In the moving frame, the momentum becomes $p'_z = 0$ in Eq. (3), and using these chiral operators (4), the Dirac Hamiltonian becomes

$$H_D = mc^2 \begin{bmatrix} 1 & 0 & 0 & -i2\sqrt{\xi}a_r \\ 0 & 1 & i2\sqrt{\xi}a_r^\dagger & 0 \\ 0 & -i2\sqrt{\xi}a_r & -1 & 0 \\ i2\sqrt{\xi}a_r^\dagger & 0 & 0 & -1 \end{bmatrix}, \quad (5)$$

where $\xi := \hbar\omega/mc^2$ is a parameter which controls the non-relativistic limit. It follows from Eq. (5), that the chiral operator couples different components of the Dirac spinor and simultaneously creates or annihilates right-handed quanta. Expressing the Dirac spinor appropriately $|\Psi\rangle := [\psi_1, \psi_2, \psi_3, \psi_4]^t$, the Hamiltonian becomes

$$\begin{aligned} H_D &= mc^2\sigma_{14}^z + g_{14}\sigma_{14}^+a_r + g_{14}^*\sigma_{14}^-a_r^\dagger \\ &+ mc^2\sigma_{23}^z + g_{23}\sigma_{23}^+a_r^\dagger + g_{23}^*\sigma_{23}^-a_r, \end{aligned} \quad (6)$$

where $g_{14} := -i2mc^2\sqrt{\xi} =: -g_{23}$ represent the coupling constants between the different spinor components. The first term in Eq. (6) which couples components $\{\psi_1, \psi_4\}$ is identical to a detuned Jaynes-Cummings interaction

$$H_{JC}^{14} = \Delta\sigma_{14}^z + (g_{14}\sigma_{14}^+a_r + g_{14}^*\sigma_{14}^-a_r^\dagger). \quad (7)$$

Likewise, the remaining term is identical to a anti-Jaynes-Cummings (AJC) interaction between $\{\psi_2, \psi_3\}$

$$H_{AJC}^{23} = \Delta\sigma_{23}^z + (g_{23}\sigma_{23}^+a_r^\dagger + g_{23}^*\sigma_{23}^-a_r), \quad (8)$$

with a similar detuning parameter $\Delta := mc^2$. Therefore, the Dirac Hamiltonian is the sum of JC and AJC terms $H_D = H_{JC}^{14} + H_{AJC}^{23}$, which is represented in Fig. 1.

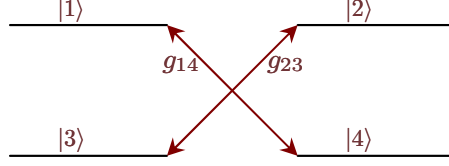


FIG. 1: Quantum Optical representation of the relativistic e^- levels coupled by means of a constant magnetic field.

This level diagram, so usual in Quantum Optics, must be interpreted as follows. According to the free Dirac equation $g_{14} = g_{23} = 0$, the spinor components $\{\psi_1, \psi_2\}$ correspond to positive energy components, while $\{\psi_3, \psi_4\}$ stand for negative energy components separated by an energy gap $\Delta\epsilon = 2mc^2$. Furthermore, these components have a well-defined value of the spin projected along the z -axis. Namely, $\{\psi_1, \psi_3\}$ are spin-up components while $\{\psi_2, \psi_4\}$ represent spin-down components. Thus, as Fig. 1 states, the interaction of a free electron with a constant magnetic field induces transitions between spin-up/spin-down and positive/negative energy components. Each transition between the large and short components $\{\psi_1, \psi_2\} \leftrightarrow \{\psi_3, \psi_4\}$ is accompanied by a spin flip and mediated through the creation or annihilation of right-handed quanta of rotation.

Taking advantage of usual methods in Quantum Optics, the whole Hilbert space can be divided into a set of invariant subspaces, which facilitate the diagonalization task. In order to do so, let us introduce the states $|j, n_r\rangle = |j\rangle|n_r\rangle$, which represent the electronic spinor component ψ_j and the electronic rotational state $|n_r\rangle := \frac{1}{\sqrt{n_r!}}(a_r^\dagger)^{n_r}|\text{vac}\rangle$. Due to the previously described mapping (6), the Hilbert space can be described as $\mathcal{H} = \tilde{\mathcal{H}} \oplus_{n_r=0}^{\infty} \mathcal{H}_{n_r}$, where $\tilde{\mathcal{H}}$ is spanned by states

$$\tilde{\mathcal{H}} = \text{span}\{|4, 0\rangle, |2, 0\rangle\}, \quad (9)$$

which have energies $\tilde{E} := \pm\Delta = \pm mc^2$ respectively. These states can be interpreted as quantum optical dark states, since they do not evolve exchanging chiral quanta (6). The remaining invariant subspaces are

$$\mathcal{H}_{n_r} = \text{span}\{|1, n_r\rangle, |4, n_r + 1\rangle, |2, n_r + 1\rangle, |3, n_r\rangle\}. \quad (10)$$

and allow a block decomposition of the Hamiltonian (5)

$$H_{n_r} = \begin{bmatrix} \Delta & -g\sqrt{n_r+1} & 0 & 0 \\ -g^*\sqrt{n_r+1} & -\Delta & 0 & 0 \\ 0 & 0 & \Delta & g\sqrt{n_r+1} \\ 0 & 0 & g^*\sqrt{n_r+1} & -\Delta \end{bmatrix}, \quad (11)$$

where $g = i2mc^2\sqrt{\xi}$ is related to the coupling constants introduced in Eq. (6). This Hamiltonian can be block-diagonalized, yielding the following energies

$$E' = \pm E'_{n_r} := \pm \sqrt{\Delta^2 + |g|^2(n_r+1)}, \quad (12)$$

which correspond to the relativistic Landau levels in Eq. (2) with $p_z = 0$. In the non-relativistic limit, where $E'_{n_r} = mc^2 + \epsilon'_{n_r}$ such that $\epsilon'_{n_r} \ll mc^2$, we find that the energy spectrum in Eq. (12) can be expressed as $\epsilon'_{n_r} \approx \hbar\omega_c(n_r+1)$, which are the usual Landau levels [7]. The associated relativistic eigenstates are

$$\begin{aligned} |\pm E'_{n_r}, 1\rangle &:= \alpha_{n_r}^{\pm} |n_r\rangle \chi_{1\uparrow} \mp i\alpha_{n_r}^{\mp} |n_r+1\rangle \chi_{2\downarrow}, \\ |\pm E'_{n_r}, 2\rangle &:= \alpha_{n_r}^{\pm} |n_r+1\rangle \chi_{1\downarrow} \mp i\alpha_{n_r}^{\mp} |n_r\rangle \chi_{2\uparrow}, \end{aligned} \quad (13)$$

where we have introduced the usual Pauli spinors

$$\chi_{1\uparrow} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \chi_{1\downarrow} := \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \chi_{2\uparrow} := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \chi_{2\downarrow} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad (14)$$

and $\alpha_{n_r}^{\pm} := \sqrt{(E'_{n_r} \pm mc^2)/2E'_{n_r}}$. The rotational and spinorial properties of the eigenstates in Eq. (13) become unavoidably entangled in the moving inertial frame \mathcal{S}' .

To obtain the corresponding solutions in the rest frame \mathcal{S} , we must perform a Lorentz boost along the OZ axis $p'^{\mu} := [E'/c, p'^x, p'^y, 0] \rightarrow p^{\mu} := [E/c, p^x, p^y, p^z]$. Considering the invariance of the four-momentum $g_{\mu\nu}p^{\mu}p^{\nu} = g_{\mu\nu}p'^{\mu}p'^{\nu}$, where the Minkowski metric tensor is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and that $p^x = p'^x, p^y = p'^y$, we come to $E'^2/c^2 = E^2/c^2 - p_z^2$. Substituting in Eq. (12)

$$E = \pm E_{n_r} := \pm \sqrt{\Delta^2 + p_z^2 c^2 + |g|^2(n_r+1)}. \quad (15)$$

These are the relativistic Landau levels in Eq. (2), whose associated eigenstates may be obtained by means of a Lorentz Boost to the Dirac spinor $\Psi(x^\mu) = S_{L_3}^{-1}\Psi'(x'^\mu)$

$$S_{L_3}^{-1} = \cosh \frac{\eta}{2} \begin{bmatrix} 1 & 0 & \tanh \frac{\eta}{2} & 0 \\ 0 & 1 & 0 & -\tanh \frac{\eta}{2} \\ \tanh \frac{\eta}{2} & 0 & 1 & 0 \\ 0 & -\tanh \frac{\eta}{2} & 0 & 1 \end{bmatrix}, \quad (16)$$

where η is the rapidity, $\cosh \eta/2 = \sqrt{(E_{n_r} + E'_{n_r})/2E'_{n_r}}$, $\tanh \eta/2 = p_z c/(E_{n_r} - E'_{n_r})$. With these expressions, one can finally obtain the eigenstates in the rest frame \mathcal{S}

$$\begin{aligned} |\pm E_{n_r}, 1\rangle &:= \alpha_{n_r}^\pm |n_r\rangle \left(\cosh \frac{\eta}{2} \chi_{1\uparrow} + \sinh \frac{\eta}{2} \chi_{2\uparrow} \right) + \\ &\quad \pm i \alpha_{n_r}^\mp |n_r + 1\rangle \left(\sinh \frac{\eta}{2} \chi_{1\downarrow} - \cosh \frac{\eta}{2} \chi_{2\downarrow} \right), \\ |\pm E_{n_r}, 2\rangle &:= \alpha_{n_r}^\pm |n_r + 1\rangle \left(\cosh \frac{\eta}{2} \chi_{1\downarrow} - \sinh \frac{\eta}{2} \chi_{2\downarrow} \right) + \\ &\quad \mp i \alpha_{n_r}^\mp |n_r\rangle \left(\sinh \frac{\eta}{2} \chi_{1\uparrow} + \cosh \frac{\eta}{2} \chi_{2\uparrow} \right), \end{aligned} \quad (17)$$

where the four spinor components get mixed in the rest frame \mathcal{S} due to the Lorentz Boost (see Fig. 2).

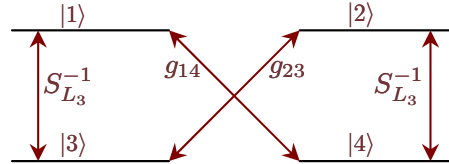


FIG. 2: Quantum Optical representation of the coupling between the relativistic levels caused by the Lorentz Boost.

Once the relativistic eigenstates have been obtained in a Quantum Optics framework, we can discuss a novel aspect of the relativistic electron dynamics, the rise of Dirac cat states. We define the notion of Dirac cat states as a coherent superposition of two mesoscopically distinct relativistic states. Our main goal now is to find the conditions which guarantee the existence of such cat states. They will turn out to be non-trivial. The mapping of the Dirac Hamiltonian (1) onto Quantum Optics Hamiltonians (6) is a key tool for finding the correct regime.

For the sake of simplicity we restrict to the regime with $p_z = 0$, where the effective dynamics of an initial state $|\Psi(0)\rangle = |z_r\rangle \chi_{1\uparrow}$, with $|z_r\rangle := e^{-\frac{1}{2}|z_r|^2} \sum_{n_r=0}^{\infty} \frac{z_r^{n_r}}{\sqrt{n_r!}} |n_r\rangle$ being a

right-handed coherent state with $z_r \in \mathbb{C}$, can be described solely by the JC-term (7). Due to the invariance of Hilbert subspaces, a blockade of the AJC term occurs (see Fig. 3), and three different regimes appear :

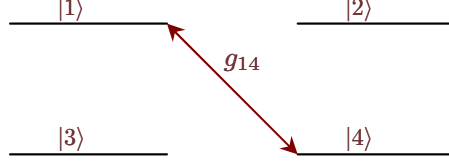


FIG. 3: Blockade of the AJC coupling

Macroscopic Regime: In this regime, the mean number of right-handed quanta $\bar{n}_r = |z_r|^2 \rightarrow \infty$, so the discreteness of the orbital degree of freedom can be neglected. Setting $z_r = i|z_r|$, the JC-term (7) can be approximately described by the semiclassical Hamiltonian

$$H_{14}^{\text{sc}} = \Delta\sigma_z + |g||z_r|(\sigma^+ + \sigma^-), \quad (18)$$

whose energies are $E^{\text{sc}} = \pm E_{z_r} := \pm\sqrt{\Delta^2 + |g|^2|z_r|^2}$. This semiclassical energy levels resemble the original spectrum (12), but the corresponding eigenvalues

$$|\pm E_{z_r}\rangle := \alpha_{z_r}^{\pm}\chi_{1\uparrow} \pm i\alpha_{z_r}^{\mp}\chi_{2\downarrow}, \quad (19)$$

with $\alpha_{z_r}^{\pm} := \sqrt{(E_{z_r} \pm \Delta)/2E_{z_r}}$, are clearly different from those in Eq.(13). In the semiclassical limit, entanglement between the spin and orbital degrees of freedom is absent. The state $|\Psi(0)\rangle := \chi_{1\uparrow}$ evolves according to

$$\begin{aligned} |\Psi(t)\rangle = & \left(\cos \Omega_{z_r}^{\text{sc}} t - \frac{i}{\sqrt{1 + 4\xi\bar{n}_r}} \sin \Omega_{z_r}^{\text{sc}} t \right) \chi_{1\uparrow} + \\ & + i \left(\sqrt{\frac{4\xi\bar{n}_r}{1 + 4\xi\bar{n}_r}} \sin \Omega_{z_r}^{\text{sc}} t \right) \chi_{2\downarrow}, \end{aligned} \quad (20)$$

where $\Omega_{z_r}^{\text{sc}} := E_{z_r}/\hbar$ is the semiclassical Rabi frequency. Therefore, Dirac cats states of the orbital degree of freedom cannot be produced during the dynamical evolution.

Microscopic Regime: In this limit, $\bar{n}_r = |z_r|^2 \lesssim 10$ is small enough for the discreteness of the orbital degree of freedom to become noticeable. Especially interesting is the evolution of the vacuum of right-handed quanta

$$\begin{aligned} |\Psi(t)\rangle = & \left(\cos \omega_0 t - \frac{i}{\sqrt{1 + 4\xi}} \sin \omega_0 t \right) |0\rangle\chi_{1\uparrow} + \\ & + \left(\sqrt{\frac{4\xi}{1 + 4\xi}} \sin \omega_0 t \right) |1\rangle\chi_{2\downarrow}, \end{aligned} \quad (21)$$

where $\omega_0 := \frac{mc^2}{\hbar} \sqrt{1+4\xi}$ is the vacuum Rabi frequency. We observe how the spinorial and orbital degrees of freedom become inevitably entangled as time evolves due to the interference of positive and negative energy solutions, i.e. *Zitterbewegung* [8]. This behavior is crucial for the generation of Schrödinger cat states, although their growth cannot occur under this regime since the orbital degree of freedom are not of a mesoscopic nature.

Mesoscopic Regime: When the mean number of orbital quanta $10 \lesssim \bar{n}_r \lesssim 100$ attains a mesoscopic value, certain collapses and revivals in the Rabi oscillations (21) occur [9]. An asymptotic approximation which accounts for the collapse-revival phenomenon has been derived in [10, 11], and its validity has been experimentally tested in Cavity QED (CQED) [12]. Below, we derive a relativistic mesoscopic approximation, which allows us to predict the generation of Dirac cat states.

Let us first discuss this asymptotic approximation, where the semiclassical eigenstates (19) play an essential role. The initial states $|\Psi^\pm(0)\rangle := |\pm E_{z_r}\rangle|z_r\rangle$ evolve according to

$$|\Psi^\pm(t)\rangle \approx \left(\alpha_{z_r}^\pm e^{\mp i \frac{|g|^2}{2\hbar E_{z_r}} t} \chi_{1\uparrow} \pm i \alpha_{z_r}^\mp \chi_{2\downarrow} \right) e^{\mp i \Theta t} |z_r\rangle, \quad (22)$$

where $\Theta := \frac{1}{\hbar} \sqrt{\Delta^2 + |g|^2 a_r^\dagger a_r}$ depends on the chiral operators. The electron spin and orbital degrees of freedom remain disentangled throughout the whole evolution $|\Psi^\pm(t)\rangle = |\Phi_{\text{sp}}^\pm(t)\rangle \otimes |\Phi_{\text{orb}}^\pm(t)\rangle$. This peculiar behavior may be compared to the *Zitterbewegung* oscillations in Eq. (21), where entanglement plays a major role.

For times shorter than the usual revival time $t \ll t_R := 2\pi E_{z_r} \hbar / |g|^2$, the asymptotic approximation in Eq.(22) can be pushed further, and a suggestive expression for the evolved orbital state $|\Phi_{\text{orb}}^\pm(t)\rangle := e^{\mp i \Theta t} |z_r\rangle$ follows

$$|\Phi_{\text{orb}}^\pm(t)\rangle \approx e^{\mp i \frac{t}{\hbar} \left(E_{z_r} - \frac{|g|^2 |z_r|^2}{2E_{z_r}} \right)} |z_r\rangle e^{\mp i \frac{|g|^2 t}{2\hbar E_{z_r}}}. \quad (23)$$

Up to an irrelevant global phase, the short time evolution of the orbital coherent state yields another coherent state whose phase evolves in time according to Eqs. (23). Considering the position operators $X = \tilde{\Delta}(a_r + a_r^\dagger + a_l + a_l^\dagger)/2$, $Y = i\tilde{\Delta}(a_r - a_r^\dagger - a_l + a_l^\dagger)/2$, we calculate the expectation value that describes the electron trajectory $\langle \mathbf{X}(\mathbf{t}) \rangle_\pm := (\langle X(t) \rangle_\pm, \langle Y(t) \rangle_\pm)$, yielding the following

$$\begin{aligned}\langle \mathbf{X}(\mathbf{t}) \rangle_+ &= \tilde{\Delta} |z_r| (-\sin \Omega_{\text{rot}} t, +\cos \Omega_{\text{rot}} t), \\ \langle \mathbf{X}(\mathbf{t}) \rangle_- &= \tilde{\Delta} |z_r| (+\sin \Omega_{\text{rot}} t, +\cos \Omega_{\text{rot}} t),\end{aligned}\tag{24}$$

where $\Omega_{\text{rot}} := |g|^2/2E_{z_r}\hbar$. Therefore solutions $|\Psi^+\rangle$ rotate counterclockwise around the z -axis, whilst $|\Psi^-\rangle$ rotate clockwise. Considering $|\Psi(0)\rangle := \chi_{1,\uparrow}|z_r\rangle = (\alpha_{z_r}^+|+E_{z_r}\rangle + \alpha_{z_r}^-|-E_{z_r}\rangle)|z_r\rangle$, which involves both semiclassical solutions (19), it splits up in two components which rotate in opposite directions as time elapses

$$|\Psi(t)\rangle = \alpha_{z_r}^+ |\Phi_{\text{sp}}^+(t)\rangle |\Phi_{\text{orb}}^+(t)\rangle + \alpha_{z_r}^- |\Phi_{\text{sp}}^-(t)\rangle |\Phi_{\text{orb}}^-(t)\rangle,\tag{25}$$

where we have introduced the spinor states for clarity

$$|\Phi_{\text{sp}}^\pm(t)\rangle := \left(\alpha_{z_r}^+ e^{\mp i \frac{|g|^2}{2\hbar E_{z_r}} t} \chi_{1\uparrow} \pm i \alpha_{z_r}^- \chi_{2\downarrow} \right).\tag{26}$$

Once we have discussed the relativistic asymptotic approximation (25), we can proceed with the generation of a relativistic version of Schrödinger cat states. In order to obtain Dirac cats, we need the following condition

$$|\Phi_{\text{sp}}^+(t_d)\rangle = e^{i\delta} |\Phi_{\text{sp}}^-(t_d)\rangle =: |\tilde{\Phi}_d\rangle,\tag{27}$$

to be fulfilled, where t_d corresponds to the Dirac cat time and $\delta \in \mathbb{R}$. If such a constraint (27) is satisfied, then the time evolution (25) under the mesoscopic regime leads to

$$|\Psi(t_d)\rangle = |\tilde{\Phi}_d\rangle (\alpha_{z_r}^+ |\Phi_{\text{orb}}^+(t)\rangle + e^{i\delta} \alpha_{z_r}^- |\Phi_{\text{orb}}^-(t)\rangle),\tag{28}$$

and we obtain a coherent superposition of states in the orbital degree of freedom. Furthermore, using the properties of unitary evolution, it follows that

$$\langle +E_{z_r} | -E_{z_r} \rangle = 0 \mapsto \langle \Phi_{\text{orb}}^+(t_d) | \Phi_{\text{orb}}^-(t_d) \rangle = 0,\tag{29}$$

and therefore the orbital state in Eq.(28)

$$|\Phi_{\text{orb}}^{\text{cat}}\rangle := \alpha_{z_r}^+ |\Phi_{\text{orb}}^+(t_d)\rangle + e^{i\delta} \alpha_{z_r}^- |\Phi_{\text{orb}}^-(t_d)\rangle,\tag{30}$$

represents a coherent superposition of mesoscopically distinct states, and consequently a Schrödinger cat in the relativistic scenario. The generation of these unusual cats is therefore

subjected to the verification of condition (27). At half revival time $t_d = t_R/2 = \pi E_{z_r} \hbar / |g|^2$, we find

$$|\langle \Phi_{\text{sp}}^+(t_d) | \Phi_{\text{sp}}^-(t_d) \rangle| \approx \sqrt{\frac{4\xi \bar{n}_r}{1 + 4\xi \bar{n}_r}}. \quad (31)$$

In order to satisfy the aforementioned constraint, one must take the ultra-relativistic limit $\xi \gg 1/\bar{n}_r$, where Eq. (31) is $|\langle \Phi_{\text{sp}}^+(t_d) | \Phi_{\text{sp}}^-(t_d) \rangle| \approx 1 + \mathcal{O}(\frac{1}{\bar{n}_r})$ of the order of unity, and thus a Dirac cat is generated. As a concluding remark, we stress the relativistic nature of these cat states. In the non-relativistic scenario Eq.(31) yields

$$|\langle \Phi_{\text{sp}}^+(t_s) | \Phi_{\text{sp}}^-(t_s) \rangle| \approx 2\sqrt{\xi \bar{n}_r} + \mathcal{O}(\xi^{3/2}) \ll 1, \quad (32)$$

and thus the cat generation condition cannot be fulfilled in this case. As the electron slows down, the coherence of (28) vanishes and the Dirac cat disappears.

In summary, we have found a novel correspondence between Quantum Optics and Relativistic Quantum Mechanics. This perspective allows an insightful derivation of the relativistic Landau levels, and reveals a wide variety of original phenomena present in the relativistic system. Remarkably, we have predicted the existence of Dirac cat states, a relativistic version of the unusual Schrödinger cat states, which have a purely relativistic nature and occur under a mesoscopic regime.

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